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On probability distributions arising
from points on a graph

by

A.R. Bloemena

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1. Introduction

Given a set of n points, numbered $1, \dots, n$, and a $n \times n$ matrix M , with elements m_{ij} , satisfying

$$(1.1) \quad m_{ij} = m_{ji} \quad (i \neq j),$$

$$(1.2) \quad m_{ii} = 0,$$

$$(1.3) \quad \text{for each } i \quad m_{ij} \neq 0 \text{ for at least one } j, \text{ and}$$

$$(1.4) \quad 0 \leq m_{ij} < \infty.$$

The set of points and the matrix M can be interpreted as a finite multigraph (cf. C. BERGE (1958), D. KOENIG (1936)), where the number of joins between point i and j is equal to m_{ij} . If $m_{ij} = 0$, this means that there is no join between i and j . Assumption (1.2) states that there are no loops. Assumption (1.3) implies that no point is isolated.

From the n points two samples are taken. We shall consider two cases.

Case I "non free sampling": from the points $1, \dots, n$ r_1 and r_2 points are chosen at random without replacement ($r_1 + r_2 \leq n$). The r_1 points will be denoted as black (B) points, the r_2 points as white (W) ones, while finally the $n - r_1 - r_2$ remaining points are the red (R) ones.

Case II "free sampling": n independent trials are performed, each trial resulting in the event B with probability p_1 , in the event W with probability p_2 , and in the event R with probability $1 - p_1 - p_2$. Point number i is allotted the colour indicated by the outcome of the i -th trial.

Consider the random variables $\underline{x}_{ij}$ and $\underline{y}_{ij}$ ($i, j = 1, \dots, n$), defined by

$$\underline{x}_{ii} = 0 \quad \text{spr } 0,$$

$$\underline{y}_{ii} = 0 \quad \text{spr } 0,$$

and for $i \neq j$

$$\underline{x}_{ij} = \begin{cases} 1 & \text{if point } i \text{ and } j \text{ are both black} \\ 0 & \text{if not.} \end{cases}$$

$$\underline{y}_{ij} = \begin{cases} 1 & \text{if point } i \text{ is black and } j \text{ is white, or} \\ & \text{point } i \text{ is white and } j \text{ is black,} \\ 0 & \text{if not.} \end{cases}$$

$$(1.5) \quad \underline{x} = \sum_{ij} m_{ij} \underline{x}_{ij},$$

$$(1.6) \quad \underline{y} = \sum_{ij} m_{ij} \underline{y}_{ij}.$$

We shall also consider a more general situation. Let be given a set of random variables $\underline{z}_{ij}$, where $\underline{z}_{ii} = 0$ spr 0,

while for $i \neq j$ $\underline{z}_{ij}$ is either 0 or 1. Define

$$(1.7) \quad \underline{z} = \sum_{ij} m_{ij} \underline{z}_{ij}.$$

In the following we shall give results on the stochastic properties of $\underline{x}$, $\underline{y}$ and $\underline{z}$. The proofs of these results will be given in a forthcoming thesis.

2. Previous work on the subject

P.A.P. MORAN (1948) considers a "statistical map", equivalent to our graph for $m_{ij} = 0$ or 1, where the points are chosen by "free" and "non free" sampling. He gives for both cases the first and second moments of the number of black-black joins (thus for $\underline{x}$) and the third and fourth moment for the case of free sampling. He proves the asymptotic normality of $\underline{x}$ and $\underline{y}$ (free sampling for a rectangular twodimensional lattice, where there are joins between neighbouring points in the direction of both axis (cf. also P.A.P. MORAN (1947)).

There exists a large number of papers on the subject by P.V. KRISHNA IYER (1948-1953), most of them in an extremely-hard-to-get journal, viz. the *Journal of The Indian Society for Agricultural Statistics*. As far as we are aware, KRISHNA IYER only deals with rectangular lattices, where neighbouring points are joined in the direction of both axis, but also diagonal joins are considered in a number of his papers. The results of KRISHNA IYER are mostly on the first four moments or cumulants, and statements about asymptotic normality.

A report by BLOEMEN and van EEDEN contains a number of exact results for rectangular lattices (non free sampling). The present report is an outgrowth of this last paper, which arose from a study of the distribution of a statistic, obtained in a psychological test.

Some older papers on the subject are by H. TODD (1940) and D.J. FINNEY (1947).

3. Some graphtheoretical notions

Consider a set S of points and a subset U of the set of all joins between these points. The combination (S, U) is usually called a graph. For a detailed treatment of theory of graphs, we refer to D. KOENIG (1936) and C. BERGE (1958).

For our purpose we use the word "graph" to denote a set of k oriented joins, labelled $J_1, \dots, J_k$, between ℓ ($2 \leq \ell \leq 2k$) points, such that no points are isolated (are not connected to at least one other point), and loops do not occur. Multiple joins are admitted.

A point to which join J_i is connected will be called the second point of J_i if the orientation of the join is towards the point; if not, it will be called the first point of J_i .

To each graph there corresponds a symmetrical $2k \times 2k$ matrix A , consisting of k^2 2×2 block-matrices A_{ij} ($i, j = 1, \dots, k$), with elements

$$\text{for } \mu, \lambda = 1, 2$$

$$a_{i\mu, i\lambda} = 0,$$

and for $i \neq j$

$$a_{i\mu, j\lambda} = \begin{cases} 1 & \text{if the } \mu\text{-th point of } J_i \text{ coincides} \\ & \text{with the } \lambda\text{-th point of } J_j, \\ 0 & \text{if not} \end{cases}$$

All graphs having the same matrix A are considered to be equivalent. E.g. both

have as matrix

$$\left(\begin{array}{cc|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 1 & \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & \end{array} \right)$$

and are therefore equivalent.

The $k \times k$ matrix with elements

$$b_{ij} = \sum_{\mu=1}^2 \sum_{\lambda=1}^2 a_{i\mu, j\lambda}$$

will be called the configurationmatrix.

Consider two graphs G_1 and G_2 , each based on ℓ points and k (labelled and oriented) joins. If G_1 and G_2 are not identical, but a permutationmatrix P exists such that for the configurationmatrices B_1 and B_2 the relation

$$B = P B_2 P^{\vee}$$

holds, we shall say that G_1 and G_2 have the same configuration.

A graph $G = (S, U)$ is called connected if from every point $i \in S$ one can reach any other point of S by travelling along the joins of the set U , neglecting the orientation of the joins. A graph which is not connected, can be decomposed in a number of connected components. This decomposition is unique (cf. D.KÖNIG, 1936, p.15). A configuration-matrix of a not connected graph (if necessary after premultiplication with a permutationmatrix P , and postmultiplication with $P^{\vee}$) is a logical sum of the configuration-matrices of each of the connected components.

A connected graph with k joins has at most $k + 1$ points. It has at least two points. For l satisfying

$$2 \leq l \leq k+1$$

finitely many, say $q_{k,l}$, distinct configurations exists corresponding to connected graphs based on k joins and l points.

Let $C_{k,l}^{(\alpha)}$ be the α -th one ($\alpha = 1, \dots, q_{k,l}$).

The configuration of a graph having h connected components ($1 \leq h \leq \lfloor \frac{l}{2} \rfloor$) can now be indicated symbolically by

$$\sum C_{k_i, l_i}^{(\alpha_i)}$$

if the i -th connected component has a configuration $C_{k_i, l_i}^{(\alpha_i)}$.

If among the h connected components g_j have the same configuration

$$C_{k_j, l_j}^{(\alpha_j)} \quad \text{we may also write} \quad \sum g_j C_{k_j, l_j}^{(\alpha_j)}$$

as the symbol for the configuration of the graph.

By means of the operator $\mathcal{N}(\quad)$, operating on the symbol of a configuration we indicate the number of distinct graphs, having this configuration. It can be proved that if $\sum_i k_i g_i = k$,

$$(3.1) \quad \mathcal{N}\left(\sum_{i=1}^s g_i C_{k_i, l_i}^{(\alpha_i)}\right) = k! \prod_{i=1}^s \frac{1}{g_i!} \left\{ \frac{\mathcal{N}(C_{k_i, l_i}^{(\alpha_i)})}{k_i!} \right\}^{g_i}.$$

The calculation of $N(C_{k_i, \ell_i}^{(\alpha_i)})$ proceeds by means of recurrence relations.

4. A general expression for the moments of $\underline{z}$

In order to calculate the k-th moment of $\underline{z}$, we have to consider products like

$$(4.1) \quad m_{i_{1,1} i_{1,2}} m_{i_{2,1} i_{2,2}} \dots m_{i_{k,1} i_{k,2}}$$

and

$$(4.2) \quad \underline{z}_{i_{1,1} i_{1,2}} \underline{z}_{i_{2,1} i_{2,2}} \dots \underline{z}_{i_{k,1} i_{k,2}},$$

where $i_{1,1}, \dots, i_{k,2}$ are, say, l different integers from the range $1, \dots, n$. To each such products there corresponds a graph. Let each of the subscripts of (4.2) correspond to a point of the graph. If two or more subscripts are equal, they correspond to a same point, thus the graph has l different points in all. Let the first subscript, $i_{j,1}$, of $\underline{z}_{i_{j,1} i_{j,2}}$ corresponds to the first

point of a join, and the second subscript, $i_{j,2}$, to the second point of the same join. We thus obtain a graph with k oriented joins and l points, no one point being isolated. We assume that always $i_{j,1} \neq i_{j,2}$ ($j=1, \dots, k$), thus no loops arise. Let the graph corresponding to (4.2) have a configuration

$$\sum_{i=1}^k C_{k_i, \ell_i}^{(\alpha_i)}$$

then the following assumption on the simultaneous distribution of the $\underline{z}_{ij}$ ($i \neq j$) is introduced:

Assumption A1

For each $k = 1, \dots$ the expectation of (4.2) does not depend on the actual value of $i_{1,1}, \dots, i_{k,2}$, but only on the configuration

$$\sum_{i=1}^k C_{k_i, \ell_i}^{(\alpha_i)}.$$

We therefore introduce the following notation for the expectation of (4.2):

$$(4.3) \quad E \left[\sum_{i=1}^k C_{k_i, \ell_i}^{(\alpha_i)} \right] \cong^{(1)} \dots \cong^{(k)}.$$

To indicate the sum over a product of k coefficients $m_{i_{j,1}, i_{j,2}} (i_{j,1} \neq i_{j,2})$, where exactly 1 out of the $2k$ subscripts are different, and in such a way, that the graph corresponding to this product has configuration $\sum_{i=1}^h C_{k_i, l_i}^{(\alpha_i)}$ we write

$$\sum_{[i_1 \neq \dots \neq i_j, i_{j+1}, \dots, i_\ell; \sum_{i=1}^h C_{k_i, l_i}^{(\alpha_i)}]} m^{(1)} \dots m^{(k)}, \quad \text{we write}$$

if the condition on the subscripts is that in the summation $i_1, \dots, i_j$ have to be different. Summation over $i_1, \dots, i_\ell$ extends from $1, \dots, n$.

Now one can derive

$$(4.4) \quad E \underline{z}^k = \sum_{\ell=2}^{2k} \sum_{h=1}^{\lfloor \frac{\ell}{2} \rfloor} \sum''_{[\sum_{i=1}^h k_i = k, \sum_{i=1}^h l_i = \ell, \alpha_i]} \mathcal{N} \left(\sum_{i=1}^h C_{k_i, l_i}^{(\alpha_i)} \right).$$

$$E \left[\sum_{i=1}^h C_{k_i, l_i}^{(\alpha_i)} \right] \underline{z}^{(1)} \dots \underline{z}^{(k)} = \sum_{[i_1 \neq \dots \neq i_\ell; \sum_{i=1}^h C_{k_i, l_i}^{(\alpha_i)}]} m^{(1)} \dots m^{(k)},$$

where $\sum''_{[\sum_{i=1}^h k_i = k, \sum_{i=1}^h l_i = \ell, \alpha_i]}$ means summation over all configurations with $\sum k_i = k$ and $\sum l_i = \ell$.

5. The moments of $\underline{x}$

a) non free sampling

Here assumption A 1 is satisfied, as

$$(5.1) \quad E \left[\sum_{i=1}^h C_{k_i, l_i}^{(\alpha_i)} \right] \underline{x}^{(1)} \dots \underline{x}^{(k)} = \frac{\binom{n}{\ell}}{\binom{n}{\ell}},$$

where $\sum l_i = \ell$.

(We omit the subscript on r_1 and p_1 , as no danger of confusion arises in the sections on $\underline{x}$.)

From (4.4) we have e.g. after some simplifications

$$E \underline{x} = \frac{n(n-1)}{n(n-1)} \sum_{i,j} m_{ij},$$

$$\begin{aligned} \sigma^2 = \underline{Ex}^2 - (\underline{Ex})^2 &= 4 \frac{r(r-1) \cdot (r-2) \cdot (n-r)}{n(n-1) \cdot (n-2) \cdot (n-3)} \sum_i \left(\sum_j m_{ij} - \frac{\sum_j m_{ij}}{n} \right)^2 \\ &+ \frac{2r(r-1) \cdot (n-r) \cdot (n-r-1)}{n^2(n-1)^2(n-2)(n-3)} \left\{ n(n-1) \sum_{ij} m_{ij}^2 - \left(\sum_{ij} m_{ij} \right)^2 \right\}. \end{aligned}$$

If $\sum_j m_{ij}$ does not depend on i , the first term of σ^2 is equal to zero. The third reduced moment and the fourth unreduced moment have been calculated as well.

b) free sampling

$$(5.2) \quad E \left[\sum_{i=1}^h C_{k_i, l_i}^{(x_i)} \right] x^{(1)} \dots x^{(k)} = p^k,$$

so e.g.

$$\begin{aligned} E \underline{x} &= p^2 \sum_{ij} m_{ij}, \\ \sigma^2 &= 2 p^2 (1-p) \left\{ (1-p) \sum_{ij} m_{ij} + 2 p \sum_{ijk} m_{ij} m_{ik} \right\}. \end{aligned}$$

6. The moments of $\underline{y}$.

Assumption A₁ is satisfied. In order to calculate

$$E \left[\sum_{i=1}^h C_{k_i, l_i}^{(x_i)} \right] y^{(1)} \dots y^{(k)}$$

we first take a point P_i of the i -th connected component ($i=1, \dots, h$) as a reference point. Colour P_i white, next all points connected by a join to P_i are coloured black, then all points connected to these black points are coloured white. If in repeating this procedure one arrives at a point which has already been given one colour, but should be coloured by the just-mentioned rule in the other colour as well, then we conclude that the i -th connected component is not bichromatic.

If no such situation arises one arrives at a stage, where all points have been allotted a colour, viz τ_i points are white and $l_i - \tau_i$ black, we then say that the i -th connected component is bichromatic.

Define

$$(6.1) \quad \mathcal{B}(\sum_i C_{k_i, \ell_i}^{(\alpha_i)}) = \begin{cases} 1 & \text{if all connected components of } \sum_i C_{k_i, \ell_i}^{(\alpha_i)} \\ & \text{are bichromatic} \\ 0 & \text{if not.} \end{cases}$$

a) non-free sampling

$$(6.2) \quad E_{[\sum_{i=1}^h C_{k_i, \ell_i}^{(\alpha_i)}]} \underline{y}^{(1)} \dots \underline{y}^{(k)} = \mathcal{B}(\sum_i C_{k_i, \ell_i}^{(\alpha_i)}) \frac{(n-l)!}{n!} \sum_{i=1}^h \sum_{\rho_i=0}^1 \frac{r_1!}{(r_1 - \sum_i (1-\rho_i)\tau_i - \sum_i \rho_i(\ell_i - \tau_i))!} \frac{r_2!}{(r_2 - \sum_i \rho_i\tau_i - \sum_i (1-\rho_i)(\ell_i - \tau_i))!}.$$

E.g.

$$\begin{aligned} E\underline{y} &= \frac{2r_1 r_2}{n(n-1)} \sum_{ij} m_{ij}, \\ \sigma^2 &= \sum_i (\sum_j m_{ij} - \frac{\sum_j m_{ij}}{n})^2 \cdot \left\{ \frac{r_1 r_2 (r_2 - 1)}{n(n-1)(n-2)} + \frac{r_2 r_1 (r_1 - 1)}{n(n-1)(n-2)} - \frac{4 r_1 r_2 (r_1 - 1)(r_2 - 1)}{n(n-1)(n-2)(n-3)} \right\} + \\ &+ 4 \sum_{ij} m_{ij}^2 \left\{ \frac{r_1 r_2}{n(n-1)} - \frac{r_1 r_2 (r_2 - 1)}{n(n-1)(n-2)} - \frac{r_2 r_1 (r_1 - 1)}{n(n-1)(n-2)} + \frac{2 r_1 r_2 (r_1 - 1)(r_2 - 1)}{n(n-1)(n-2)(n-3)} \right\} + \\ &+ 4 (\sum_{ij} m_{ij})^2 \frac{r_1 r_2 n (r_1 + r_2 + 3) - r_1 r_2 (2 r_1 r_2 + r_1 + r_2 + 2)}{n^2 (n-1)^2 (n-2)(n-3)}. \end{aligned}$$

b) free sampling

$$(6.3) \quad E_{[\sum_{i=1}^h C_{k_i, \ell_i}^{(\alpha_i)}]} \underline{y}^{(1)} \dots \underline{y}^{(k)} = \mathcal{B}(\sum_i C_{k_i, \ell_i}^{(\alpha_i)}) \frac{1}{l!} (p_1^{\tau_i} p_2^{\ell_i - \tau_i} + p_1^{\ell_i - \tau_i} p_2^{\tau_i}).$$

E.g.

$$\begin{aligned} E\underline{y} &= 2p_1 p_2 \sum_{ij} m_{ij}, \\ \sigma^2 &= 4p_1 p_2 (p_1 + p_2 - 4p_1 p_2) \sum_{ijk} m_{ij} m_{ik} + 4p_1 p_2 (1 - p_1 - p_2 + 2p_1 p_2) \sum_{ij} m_{ij}^2. \end{aligned}$$

7. Tendency towards the normal distribution

The following theorem can be proved.

Theorem 7.1

$$(7.1) \quad \left. \begin{aligned} &\text{If} \\ &\sum_j m_{ij}^a = m_a, \\ &m_a < \infty, \\ &m_a \text{ independent of } n, \end{aligned} \right\} \text{for all } a = 1, \dots,$$

and if r and n tend to infinity such that

$$\lim \frac{r}{n} = 0, \text{ with } 0 < \delta < 1,$$

then in the non-free sampling case the distribution of

$$\frac{\underline{x} - E\underline{x}}{\sigma}$$

tends to the standard normal one. $E\underline{x}$ and σ^2 have been given in section

If (7.1) is satisfied and n tends to infinity and p tends to a limit p^π ($0 < p^\pi < 1$) then in the case of free sampling the distribution of

$$\frac{\underline{x} - E\underline{x}}{\sigma}$$

tends to the standard normal one. $E\underline{x}$ and σ^2 have been given in section 5b.

If (7.1) is satisfied, and if r_1 , r_2 and n tend to infinity such that

$$\left. \begin{aligned} \lim \frac{r_1}{n} &= \delta_1 \\ \lim \frac{r_2}{n} &= \delta_2 \end{aligned} \right\} \quad \begin{aligned} 0 < \delta_1, \quad 0 < \delta_2, \\ \delta_1 + \delta_2 < 1, \end{aligned}$$

then the distribution of

$$\frac{\underline{y} - E\underline{y}}{\sigma}$$

tends to the standard normal one. $E\underline{y}$ and σ^2 have been given in section 6a.

If (7.1) is satisfied, and if n tends to infinity and r_1 and p_2 tend to finite limits p_1^π , and p_2^π ($0 < p_1^\pi, 0 < p_2^\pi, p_1^\pi + p_2^\pi < 1$) then the distribution of

$$\frac{\underline{y} - E\underline{y}}{\sigma}$$

tends to the standard normal one. $E\underline{y}$ and σ^2 have been given in section 6b.

8. Tendency towards the compound POISSON distribution

A theorem on the tendency towards the compound POISSON distribution has been proved for $\underline{z}$. The asymptotic behaviour of $\underline{x}$ and $\underline{y}$ for both the free and the non-free sampling case can be considered as a special case of this theorem. First we introduce some assumptions.

Assumption A1

(section 4) is considered to be satisfied.

Assumption A2

Consider an event $\mathcal{Z} : \underline{z}^{(1)} = 1 \cap \dots \cap \underline{z}^{(k)} = 1$, where the subscripts of the $\underline{z}$'s correspond to $\sum_{i=1}^h C_{k_i, \ell_i}^{(\alpha_i)}$.

Consider the event $\mathcal{Z} \cap \underline{z}_{ij}=1$, where the configuration is now

$$\sum_{i=2}^h C_{k_i, \ell_i}^{(\alpha_i)} + C_{k_1+\ell_1, \ell_1+1}^{(\beta)}.$$

Then if $P[\mathcal{Z}] > 0$

$$P[\underline{z}_{ij}=1 | \mathcal{Z}] \leq P[\underline{z}_{ij}=1 | \underline{z}_{i1}=1].$$

This assumption is supposed to be satisfied for $k=1, \dots$

Assumption A3

If in a configuration $\sum_{i=1}^h C_{k_i, \ell_i}^{(\alpha_i)}$, with $\sum_{i=1}^h k_i = k$, $\sum_{i=1}^h \ell_i = 1$, a point is made to coincide with another one to which it is not connected by a join, thus giving rise to a configuration

$$\sum_{i=1}^{h'} C_{k'_i, \ell'_i}^{(\alpha'_i)}, \text{ with } \sum k'_i = k, \sum \ell'_i = \ell - 1.$$

then if

$$E \left[\sum_{i=1}^{h'} C_{k'_i, \ell'_i}^{(\alpha'_i)} \right] \geq \underline{z}^{(1)} \dots \underline{z}^{(k)} > 0$$

also

$$E \left[\sum_{i=1}^h C_{k_i, \ell_i}^{(\alpha_i)} \right] \geq \underline{z}^{(1)} \dots \underline{z}^{(k)} > 0.$$

Assumption A4

If for a configuration $C_{k, \ell}^{(\alpha)}$

$$E[C_{k, \ell}^{(\alpha)}] \underline{z}^{(1)} \dots \underline{z}^{(k)} = 0$$

then one can find joins, such that if they are removed from

$C_{k, 1}^{(\alpha)}$, a configuration $C_{k', 1}^{(\beta)}$ remains with

$$E[C_{k', 1}^{(\beta)}] \underline{z}^{(1)} \dots \underline{z}^{(k')} > 0$$

We now state

Theorem 8.1

Let $\underline{z}$ be defined by (1.7), where m_{ij} ($i, j=1, \dots, n$) satisfies (1.1) through (1.4) and let assumptions $A_1, \dots, A_4$ be satisfied.

If n tends to infinity and if for every i, j, μ, λ , with $i \neq j$, μ, λ and $\mu \neq \lambda$

$$\lim P [Z_{ij} = 1 \mid Z_{\mu\lambda} = 1] = 0,$$

such that

$$(8.1) \quad \lim E Z = 2\lambda, \quad 0 < \lambda < \infty,$$

and

$$(8.2) \quad \lim E [C_{k, \frac{k}{2}}^{(\alpha)}] Z^{(1)} \dots Z^{(k)} = \sum_{i_1, \dots, i_{k+1}} [i_1, \dots, i_{k+1}; C_{k, k+1}^{(\alpha)}] m \dots m = 0$$

for all $k = 2, \dots$, and all α ,

$$(8.3) \quad \lim E [C_{k_i, l_i}^{(\alpha)}] Z^{(1)} \dots Z^{(k)} = \lim_{l \rightarrow \infty} \frac{l}{l!} E [C_{k_i, l_i}^{(\alpha)}] Z^{(k_1 + \dots + k_{i-1} + 1)} \dots Z^{(k_1 + \dots + k_i)},$$

(where $\sum_i k_i = k, \sum_i l_i = 1$) for all $h = 2, \dots, [\frac{l}{2}]$, $l = 3, \dots, 2k$ and all $k = 2, \dots$,

$$(8.4) \quad \lim \frac{\sum_{ij} m_{ij}^h}{\sum_{ij} m_{ij}} = m_h^*, \quad m_h^* < \infty, \quad h = 2, \dots \quad (m_1^* = 1),$$

and finally,

(8.5) $m_{ij} < M$ for all i and j , where M does not depend on n , then

$$(8.6) \quad \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \lim E (Z^k) = \exp \left\{ \lambda \sum_{i=1}^{\infty} m_i^* \frac{\lambda^i}{i!} \right\} - 1,$$

which is the moment-generating function of the component POISSON distribution.

If m_{ij} is either 0 or 1, and if assumptions A_1, A_2, A_3, A_4 , are satisfied, together with (8.1), (8.2), and (8.3), then $\frac{1}{2} Z$ has asymptotically a POISSON distribution with parameter λ .

Assumption (8.2) is satisfied e.g. when $\sum_j m_{ij} = m_1$, independent of i , and

$$(8.2') \quad \lim \frac{E [C_{k, \frac{k}{2}}^{(\alpha)}] Z^{(1)} \dots Z^{(k)}}{E [C_{1,2}^{(\alpha)}] Z^{(1)}} m_1^k = 0 \quad \text{for all } k \text{ and } \alpha.$$

E.g. when m_1 is a constant independent of n then (8.2') and therefore (8.2) is satisfied.

First special case of Theorem (8.1)

non free sampling

If r and n tend to infinity such that

$$\lim \frac{r}{n} = 0,$$

$$(8.7) \lim E \underline{x} = 2\lambda, \quad 0 < \lambda < \infty,$$

$$(8.8) \lim \left(\frac{r}{n} \right)^{k+1} \sum_{i_1, \dots, i_{k+1}} C_{k,k+1}^{(\alpha)} m^{(1)} \dots m^{(k)} = 0$$

for each $k=2, \dots$, and all α ,
if moreover

free sampling

If n tends to infinity and p tends to zero such that

$$\lim E \underline{x} = 2\lambda, \quad 0 < \lambda < \infty,$$

$$\lim p^{k+1} \sum_{i_1, \dots, i_{k+1}} C_{k,k+1}^{(\alpha)} m^{(1)} \dots m^{(k)} = 0$$

for all $k=2, \dots$ and all α ,
if moreover

$$\lim \frac{\sum_{ij} m_{ij}^h}{\sum_{ij} m_{ij}} = m_h^{\#}, \quad m_h^{\#} < \infty, \quad h=1, \dots \quad (m_1^{\#} = 1),$$

and finally,

$$m_{ij} < \infty \text{ for all } i \text{ and } j, \\ \text{it does not depend on } n,$$

then

$$\sum_{k=1}^{\infty} \frac{1}{k!} \lim E \left(\frac{1}{n} \underline{x} \right)^k = \exp \left\{ \lambda \sum_{i=1}^{\infty} m_i^{\#} \frac{1}{i!} \right\} - 1.$$

If $m_{ij} = 0$ or 1 and if assumptions (8.7) and (8.8) are satisfied $\frac{1}{n} \underline{x}$ has a POISSON-distribution with parameter λ .

Assumption (8.8) is satisfied when $\sum_j m_{ij} = m_i$ independent of i and $\lim \frac{r}{n} m_1 = 0$

Second special case of Theorem (8.2)

non free sampling

If r_1, r_2 and n tend to infinity such that

$$\lim \frac{r_1}{n} = 0$$

$$\lim \frac{r_2}{n} = 0,$$

$$(8.9) \lim E \underline{y} = 2\lambda, \quad 0 < \lambda < \infty,$$

free sampling

If n tends to infinity and p_1 and p_2 tend to zero, such that

$$\lim E \underline{y} = 2\lambda, \quad 0 < \lambda < \infty,$$

and:

$$(8.10) \lim E [C_{k,k+1}^{(\alpha)}] \frac{1}{n} \dots \frac{1}{n} \sum_{i_1, \dots, i_{k+1}} C_{k,k+1}^{(\alpha)} m^{(1)} \dots m^{(k)} = 0$$

for all $k=2, \dots$ and all α ,

and if

$$\lim \frac{\sum_{i,j} m_{ij}^h}{\sum_{i,j} m_{ij}} = m_h^*, \quad m_h^* < \infty, \quad h = 2, \dots \quad (m_1^* = 1),$$

and finally,

$m_{ij} < M$, for all i and j , where M is independent of n ,

then

$$\sum_{k=1}^{\infty} \frac{Z^k}{k!} \lim E \left(\frac{1}{2} \underline{y} \right)^k = \exp \left\{ \lambda \sum_{i=1}^{\infty} m_i^* \frac{Z^i}{i!} \right\} - 1.$$

If assumptions (8.9) and (8.10) are satisfied, and if m_{ij} is either 0 or 1 then $\frac{1}{2} \underline{y}$ has asymptotically $\tilde{P}$ POISSON-distribution with parameter λ .

Assumption (8.10) is satisfied e.g. when $\sum_j m_{ij} = m_1$, independent of i and

$$\lim p_1 m_1 = 0,$$

$$\lim p_2 m_1 = 0.$$

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